Some Basic Math Tools/Concepts

- **Sigma (summation) notation**
- **Sums of deviations, squared deviations and products of deviations**
- **Elasticities**
- **Functional forms (linear, polynomial, logarithmic, exponential): slopes and elasticities**

Sigma (summation) notation

1. Consider a set of n numbers $\{x_1, x_2, \ldots x_n\} = \{x_i\}_{i=1}^{n} = \{x_i\}$. Then

   $\sum_{i=1}^{n} x_i = \{x_1 + x_2 + \ldots + x_n\} = \sum x_i$.

   a. The average of the values: $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\left(x_1 + x_2 + \ldots + x_n\right)$ … note that $\sum_{i=1}^{n} x_i = n\overline{x}$.

   b. Properties:

      i. $\sum_{i=1}^{n} c = nc$

      ii. $\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i = cn\overline{x}$

      iii. $\sum_{i=1}^{n}\left(ax_i + by_i\right) = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i = an\overline{x} + bn\overline{y}$

Sums of deviations, squared deviations and products of deviations

2. Define the deviation from average/mean… $d_i = x_i - \overline{x}$. Then:

   a. Sum of deviations: $\sum_{i=1}^{n} d_i = 0$

   $$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n}\left(x_i - \overline{x}\right) = \sum_{i=1}^{n} x_i - n\overline{x} = n\overline{x} - n\overline{x} = 0$$

b. Sum squared deviations: $\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n}\left(x_i^2\right) - n\overline{x}^2$

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2 = \sum_{i=1}^{n}\left(x_i^2 - 2\overline{x}x_i + \overline{x}^2\right) = \sum_{i=1}^{n}\left(x_i^2\right) - 2\overline{x}\sum_{i=1}^{n} x_i + n\overline{x}^2$$
$$= \sum_{i=1}^{n}\left(x_i^2\right) - 2\overline{x}n\overline{x} + n\overline{x}^2 = \sum_{i=1}^{n}\left(x_i^2\right) - n\overline{x}^2$$

c. Sum of products of deviations: $\sum_{i=1}^{n}\left(x_i - \overline{x}\right)\left(y_i - \overline{y}\right) = \sum_{i=1}^{n}\left(x_i y_i\right) - n\overline{xy}$

$$\sum_{i=1}^{n}\left(x_i - \overline{x}\right)\left(y_i - \overline{y}\right) = \sum_{i=1}^{n}\left(x_i y_i - \overline{x}y_i - \overline{y}x_i + \overline{xy}\right) = \sum_{i=1}^{n}\left(x_i y_i\right) - \overline{x}n\overline{y} - \overline{y}n\overline{x} + n\overline{xy}$$
$$= \sum_{i=1}^{n}\left(x_i y_i\right) - n\overline{xy}$$

Elasticities

3. Unit free measure of responsiveness
   a. Consider $y = f(x)$
   b. Discrete changes, $\varepsilon = \frac{\%\Delta y}{\%\Delta x} = \frac{\Delta y / y}{\Delta x / x} = \frac{x}{y}\frac{\Delta y}{\Delta x}$
   c. At the margin (for incremental changes) $\varepsilon = \frac{x}{y}\frac{dy}{dx}$ (sometimes called point elasticities)

Functional forms

4. Linear functions
   a. of one variable: $y = \beta_0 + \beta_1 x$ (y-intercept: β_0)
      i. Slope: $\frac{dy}{dx} = \beta_1$
      ii. Elasticity (at a point): $\varepsilon = \frac{x}{y}\frac{dy}{dx} = \frac{\beta_1 x}{y}$

b. of many variables: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... = \beta_0 + \sum \beta_i x_i$ (y-intercept: β_0)

i. Slopes (partial derivatives): $\frac{\partial y}{\partial x_1} = \beta_1$, $\frac{\partial y}{\partial x_2} = \beta_2$ $\cdots$

ii. Elasticities (at a point): $\varepsilon_1 = \frac{x_1}{y}\frac{\partial y}{\partial x_1} = \frac{\beta_1 x_1}{y}$, $\varepsilon_2 = \frac{x_2}{y}\frac{\partial y}{\partial x_2} = \frac{\beta_2 x_2}{y}$, $\ldots$

iii. Note that if $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$, then dividing both sides by y, we have $1 = \frac{\beta_0}{y} + \frac{\beta_1 x_1}{y} + \frac{\beta_2 x_2}{y} = \frac{\beta_0}{y} + \varepsilon_1 + \varepsilon_2$ $\ldots$ so in a sense, these elasticities are like shares.[1]

5. Quadratic (2nd order polynomial): $y = \beta_0 + \beta_1 x + \beta_2 x^2$ (y-intercept: β_0)

a. Slope: $\frac{dy}{dx} = \beta_1 + 2\beta_2 x$

b. Elasticity: $\frac{x}{y}\frac{dy}{dx} = \frac{\beta_1 x + 2\beta_2 x^2}{y}$

c. Curvature: $\frac{d^2 y}{dx^2} = 2\beta_2$ (<0: concave; >0: convex)

d. Note that $\frac{dy}{dx} = \beta_1 + 2\beta_2 x^* = 0 \rightarrow x^* = -\frac{\beta_1}{2\beta_2}$ (max/min point)

6. (Natural) Logarithmic: $y = \ln(x)$ (defined for x > 0; no y intercept)

a. Slope: $\frac{dy}{dx} = \frac{1}{x}$

b. Elasticity: $\varepsilon = \frac{x}{y}\frac{dy}{dx} = \frac{1}{y}$

c. $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$, $\ln\left(\frac{x_1}{x_2}\right) = \ln(x_1) - \ln(x_2)$, and $\ln(x^\beta) = \beta \ln(x)$

d. Note that if $y = \beta_0 x_1^{\beta_1}$, then $\ln(y) = \ln(\beta_0) + \beta_1 \ln(x_1)$, and the (constant) elasticity (of y wrt x) is: $\frac{x}{y}\frac{dy}{dx} = \frac{x}{y}\beta_0 \beta_1 x_1^{\beta_1 - 1} = \frac{x}{y}\beta_1 \frac{y}{x} = \beta_1$.

[1] They are shares if $\beta_0 = 0$.

Some Basic Math Concepts/Tools

7. Exponential: $y = e^{\beta_1 x}$ (y intercept is 1)

    a. Inverse of the ln(.) function, so $x = e^{\ln(x)}$ *and* $x = \ln(e^x)$

    b. Slope: $y = e^{\beta_1 x} \rightarrow \frac{dy}{dx} = \beta_1 e^{\beta_1 x}$

    c. Elasticity: $\varepsilon = \frac{x}{y}\frac{dy}{dx} = \frac{x\beta_1 e^{\beta_1 x}}{y} = \beta_1 x$

    d. $y = e^{\beta_0 + \beta_1 x_1 + \beta_2 x_2} = e^{\beta_0} e^{\beta_1 x_1} e^{\beta_2 x_2}$ *and* $\ln(y) = \ln(e^{\beta_0}) + \ln(e^{\beta_1 x_1}) + \ln(e^{\beta_2 x_2}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$
    … so ln(y) is linear in the x's.